Markscheme

May 2023

Mathematics: analysis and approaches

Higher level

Paper 2

29 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	$5.65685...$ *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word "their" in a description, to indicate that candidates may be using an incorrect value.
- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and 1,9 or 1000 and 1,000 and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

Section A

1. (a) $a = 1.93258...$, $b = 7.21662...$

$a = 1.93$, $b = 7.22$ ***A1A1***

[2 marks]

(b) attempt to substitute $d = 20$ into their equation ***(M1)***

$\text{height} = 45.8683...$

$\text{height} = 45.9$ (cm) ***A1***

[2 marks]

Total [4 marks]

2. **METHOD 1**

attempt to substitute into cosine rule **_(M1)_**

$154^2 = 150^2 + 90^2 - 2(150)(90)\cos \text{A}\hat{\text{P}}\text{B}$ OR $\cos \text{A}\hat{\text{P}}\text{B} = \dfrac{150^2 + 90^2 - 154^2}{2(150)(90)}$ **_(A1)_**

$\text{A}\hat{\text{P}}\text{B} = 75.2286...°$ OR $1.31298...$ radians

$\text{A}\hat{\text{P}}\text{B} = 75.2°$ OR 1.31radians **_(A1)_**

valid approach to find θ **_(M1)_**

$\theta = \dfrac{180° - \text{A}\hat{\text{P}}\text{B}}{2}$ OR $\theta = \dfrac{180° - 75.2286...°}{2}$ $(= 52.3856...)$ OR

$\theta = \dfrac{\pi - 1.31298...}{2}(= 0.914302...)$

valid approach to express h in terms of θ **_(M1)_**

$\sin\theta = \dfrac{h}{150}$ OR $h = 150\sin 52.3856...°$

$h = 118.820...$

$h = 119$ (m) **_A1_**

[6 marks]

METHOD 2

attempts to find either the distance between the buildings or the difference in height between the buildings in terms of θ **_(M1)_**

distance between the buildings is $(150 + 90)\cos\theta$ and the difference in height between the buildings is $(150 - 90)\sin\theta$ **_(A1)_**

uses Pythagoras and attempts to solve for θ **_(M1)_**

$(60\sin\theta)^2 + (240\cos\theta)^2 = 154^2$

$\theta = 0.914302...\ (= 52.3856...°)$ **_(A1)_**

$\dfrac{h}{150} = \sin(0.914302...)$ **_(M1)_**

$h = 118.820...$

$h = 119$ (m) **_A1_**

[6 marks]

3. (a) evidence of attempting to find correct area under normal curve ***(M1)***

$\mathrm{P}(W > 210)$ OR sketch

$\mathrm{P}(W > 210) = 0.115069...$

$\mathrm{P}(W > 210) = 0.115$ ***A1***

[2 marks]

(b) recognizing $\mathrm{P}(W < w) = 1 - \mathrm{P}(w < W < 210) - \mathrm{P}(W > 210)$ ***(M1)***

$\mathrm{P}(W < w) = 1 - 0.8 - 0.115069...$

$\mathrm{P}(W < w) = 0.084930...$

$\mathrm{P}(W < w) = 0.0849$ ***A1***

[2 marks]

(c) evidence of attempting to use inverse normal function ***(M1)***

$w = 197.136...$

$w = 197$ (grams) ***A1***

[2 marks]

(d) recognition of binomial distribution ***(M1)***

$X \sim \mathrm{B}(10, 0.0849302...)$

$\mathrm{P}(X = 1) = 0.382076...$

$\mathrm{P}(X = 1) = 0.382$ ***A1***

[2 marks]

Total [8 marks]

4. attempt to use the binomial expansion of $(x+h)^8$ ***(M1)***

$^8C_0x^8h^0 + {}^8C_1x^7h^1 + {}^8C_2x^6h^2 + \ldots$

$a = 8h$ (accept $^8C_1\,h$) ***A1***

$b = 28h^2$ (accept $^8C_2\,h^2$) ***A1***

$d = 70h^4$ (accept $^8C_4\,h^4$) ***A1***

recognition that there is a common ratio between their terms ***(M1)***

$8h \times r = 28h^2$ OR $28h^2 \times r = 70h^4$ OR $8h \times r^2 = 70h^4$

correct equation in terms of h ***(A1)***

$\frac{28h^2}{8h} = \frac{70h^4}{28h^2}$ (or equivalent)

$h = 1.4$ ***A1***

[7 marks]

5. (a) recognize that acceleration is zero when $v'(t) = 0$ OR at a local maximum on the graph of v ***(M1)***

$t_1 = 0.394791...$

$t_1 = 0.395\left(= \arctan\left(\frac{5}{12}\right)\right)$ (seconds) ***A1***

[2 marks]

(b) recognition that $v = 0$ ***(M1)***

sketch OR $t = 4.71238...$ OR $t = 10.9955...$

$t_2 = 10.9955...$

$t_2 = 11.0\left(= \frac{7\pi}{2}\right)$ ***A1***

[2 marks]

(c) $\int_{t_1}^{t_2} |v|\, \mathrm{d}t$ OR $\int_{0.394791...}^{10.9955...} |v|\, \mathrm{d}t$ OR $\int_{0.394791...}^{4.71238...} v\, \mathrm{d}t + \int_{4.71238...}^{10.9955...} |v|\, \mathrm{d}t\left(= 6.53806... + 1.29313...\right)$

OR $\int_{0.394791...}^{4.71238...} v\, \mathrm{d}t - \int_{4.71238...}^{10.9955...} v\, \mathrm{d}t\left(= 6.53806... - (-1.29313...)\right)$ ***(A1)***

distance = $7.83118\ldots$

= 7.83 (m) ***A1***

[2 marks]

Total [6 marks]

6. (a) **METHOD 1**

the general point on L has coordinates $(\lambda, 2-2\lambda, 4-2\lambda)$

substitutes this general point into both Π_1 and Π_2 ***(M1)***

$2\lambda-(2-2\lambda)+2(4-2\lambda)(=2\lambda-2+2\lambda+8-4\lambda)$ ***A1***

$=6$ ***AG***

$4\lambda+3(2-2\lambda)-(4-2\lambda)(=4\lambda+6-6\lambda-4+2\lambda)$ ***A1***

$=2$ ***AG***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

Note: Award ***(M1)A0A0*** for correct verification using a specific value of λ.

continued…

Question 6 continued

METHOD 2

substitutes $(0,2,4)$ into both Π_1 and Π_2 and shows that

$0-2+8=6$ and $0+6-4=2$ ***A1***

hence $(0,2,4)$ lies in both Π_1 and Π_2 ***AG***

EITHER

attempts to find $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ ***M1***

$= \begin{pmatrix} -5 \\ 10 \\ 10 \end{pmatrix}$ ***A1***

OR

attempts to find $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \bullet \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix}$ ***M1***

$(2+2-4)=0$ and $(4-6+2)=0$ ***A1***

THEN

(so $\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ is perpendicular to both normal vectors)

so the vector equation of L can be written as $\boldsymbol{r} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ ***AG***

Note: Award ***M1*** for substituting $x=0$ (or $y=2$ or $z=4$) into Π_1 and Π_2 and solving simultaneously, for example, solving $-y+2z=6$ and $3y-z=2$. Award ***A1*** for $y=2$ and $z=4$, for example.

continued…

Question 6 continued

METHOD 3

attempts row reduction to obtain *eg*,

$x+\frac{z}{2}=2$ and $y-z=-2$ ***(M1)***

substitutes $x=\lambda$ into $x+\frac{z}{2}=2$, solves for z and obtains $z=4-2\lambda$ ***A1***

substitutes $z=4-2\lambda$ into $y-z=-2$, solves for y and obtains $y=2-2\lambda$ ***A1***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

METHOD 4

attempts to solve $2x-y+2z=6$ and $4x+3y-z=2$ ***(M1)***

for example, $x=\lambda, y=2-2\lambda, z=4-2\lambda$ ***A2***

so the vector equation of L can be written as $\boldsymbol{r}=\begin{pmatrix}0\\2\\4\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\-2\end{pmatrix}$ ***AG***

Note: Only award marks for convincing use of a GDC.

[3 marks]

continued…

Question 6 continued

(b) **EITHER**

the position vector for point P nearest to the origin is perpendicular to the direction of L

$$\begin{pmatrix}\lambda \\ 2-2\lambda \\ 4-2\lambda\end{pmatrix}\cdot\begin{pmatrix}1 \\ -2 \\ -2\end{pmatrix}=0$$ ***(M1)***

$\lambda-2(2-2\lambda)-2(4-2\lambda)=0$ ***(A1)***

$9\lambda-12=0$ ***(A1)***

OR

let s be the distance from the origin to a point P on L, then

$s^2=\lambda^2+(2-2\lambda)^2+(4-2\lambda)^2$ ***(A1)***

attempts to find λ such that $\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda}=0$ ***(M1)***

either $\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda}=18\lambda-24(=0)$ or a graph of s^2 versus λ ***(A1)***

Note: Award as above for use of $s=\sqrt{\lambda^2+(2-2\lambda)^2+(4-2\lambda)^2}$.

THEN

$\lambda=\frac{4}{3}$ ***A1***

$\mathrm{P}\left(\frac{4}{3},-\frac{2}{3},\frac{4}{3}\right)\left(\mathrm{P}(1.33,-0.667,1.33)\right)$ ***A1***

[5 marks]

Total [8 marks]

7. attempts to express x in terms of $\tan y$ ***(M1)***

$x = \tan y + 2$ ***(A1)***

let V be the volume of the solid

correctly uses $V = \pi\int_a^b x^2 \mathrm{d}y$ ***(M1)***

Note: Award ***M0*** for $V = \pi\int_a^b \left(\arctan(x-2)\right)^2 dy$

$V = \pi\int_0^{\frac{\pi}{3}} \left(\tan y + 2\right)^2 \mathrm{d}y \left(= \pi\int_0^{1.0472...} \left(\tan y + 2\right)^2 \mathrm{d}y\right)$ ***(A1)***

$= 24.0213...$

$= 24.0 \ \left(= \pi\left(4\ln 2 + \pi + \sqrt{3}\right)\right)$ ***A1***

Note: GDC in degrees gives13.3

[5 marks]

8. (a) **EITHER**

attempts to find the y- coordinate of either the local minimum point or the local maximum point ***(M1)***

OR

attempts to find the discriminant of $2x-5=y\left(x^2-3\right)$ $\left(yx^2-2x+(5-3y)=0\right)$ ***(M1)***

$\Delta=4-4y\left(5-3y\right)\left(=4-20y+12y^2\right)$

THEN

$y=1.43425...$ (local min.) and $y=0.232408...$ (local max.) ***(A1)(A1)***

$g\left(x\right)\le 0.232$ OR $g\left(x\right)\ge 1.43$ ($g\left(x\right)\le \frac{-\sqrt{13}+5}{6}$ OR $g\left(x\right)\ge \frac{\sqrt{13}+5}{6}$) ***A1***

Note: Accept other valid notations such as interval notation.

[4 marks]

(b) $\frac{2\left|x\right|-5}{x^2-3}\ge 0$ (since $\cos t<0$ for $\frac{\pi}{2}<t\le\pi$) ***(R1)***

attempts to solve graphically or algebraically ***(M1)***

$x\le-\frac{5}{2}$ OR $-\sqrt{3}<x<\sqrt{3}$ $\left(-1.73<x<1.73\right)$ OR $x\ge\frac{5}{2}$ ***A1***

[3 marks]

Total [7 marks]

9. **METHOD 1**

10 numbers of the form $3n$, 10 numbers of the form $(3n-1)$ and 10 numbers of the form $(3n-2)$ (may be seen anywhere) ***(M1)***

considers one of the following two cases of forming a sum divisible by 3 ***(M1)***

case 1:

chooses 3 numbers of the form $3n$ or chooses 3 numbers of the form $(3n-1)$ or chooses 3 numbers of the form $(3n-2)$

${}^{10}C_3 + {}^{10}C_3 + {}^{10}C_3 \ \left(= 3 \times {}^{10}C_3 = 3 \times 120 = 360\right)$ ways ***A1***

case 2:

chooses 1 number of the form $3n$ and chooses 1 number of the form $(3n-1)$ and chooses 1 number of the form $(3n-2)$

${}^{10}C_1 \times {}^{10}C_1 \times {}^{10}C_1 \ \left(= \left({}^{10}C_1\right)^3 = 10^3 = 1000\right)$ ways OR $\dfrac{{}^{30}C_1 \times {}^{20}C_1 \times {}^{10}C_1}{3!}(=1000)$ ways ***A1***

total number of ways is $3 \times {}^{10}C_3 + {}^{10}C_1 \times {}^{10}C_1 \times {}^{10}C_1 \ (= 360 + 1000)$

$= 1360$ ***A1***

METHOD 2

total number of ways of choosing 3 numbers (without restriction) is ${}^{30}C_3 = 4060$ ***A1***

attempts to find the total number of ways of choosing 3 numbers whose sum is not divisible by 3 ***(M1)***

chooses 2 numbers from one group and chooses 1 number from another group

eg chooses 2 numbers of the form $3n$ and chooses 1 number of the form $3n-1$

$3! \times {}^{10}C_2 \times {}^{10}C_1 = 2700$ ***(M1)A1***

Note: Award ***(M1)*** for any integer multiple of ${}^{10}C_2 \times {}^{10}C_1$.

total number of ways is $4060 - 2700$

$= 1360$ ***A1***

[5 marks]

Section B

10. (a) $7.8 = \frac{2\pi}{\text{period}}$ ***(M1)***

$\frac{2\pi}{7.8} = 0.805536...$

$\text{period} = 0.806\left(= \frac{20\pi}{78}\right)$ ***A1***

[2 marks]

(b) **METHOD 1**

(i) $\text{amplitude} = \frac{\max - \min}{2}$ ***(M1)***

$\frac{1.8-1}{2}$

$a = -0.4$ ***A1***

(ii) $b = 1.4$ ***A1***

METHOD 2

attempt to form two simultaneous equations in a and b ***(M1)***

$H(0) = 1 \Rightarrow a + b = 1$, $H\left(\frac{\pi}{7.8}\right) = 1.8 \Rightarrow -a + b = 1.8$

$a = -0.4, b = 1.4$ ***A1A1***

[3 marks]

continued…

Question 10 continued

(c) **EITHER**

$\frac{5}{\text{period}} = 6.207... < 6\frac{1}{2}$ ***(A1)***

OR

consideration of number of maximums on graph in first 5 seconds ***(A1)***

OR

maximums when $t = 0.403, 1.21, 2.01, 2.82, 3.62, 4.43$ ***(A1)***

THEN

6 times ***A1***

[2 marks]

(d) recognizing that $H(t) = 1.5$ ***(M1)***

$-0.4\cos(7.8t) + 1.4 = 1.5$

$0.233779...$

$t = 0.234$ (seconds) ***A1***

[2 marks]

continued…

Question 10 continued

(e) finding second time height is 1.5 metres ***(M1)***

$t = 0.571757...$

in each period, height is greater than 1.5 metres for 0.337978… seconds ***(A1)***

Note: Award ***(M1)(A1)*** for total time $2.02787...$ seen.

multiplying their value by 6 and divide by 5 ***(M1)***

$\frac{0.337978...\times 6}{5}$ OR $\frac{2.02787...}{5}$

$= 0.405574...$

$\text{P(height is greater than 1.5 m)} = 0.406$ ***A1***

[4 marks]

Total [13 marks]

11. (a) attempts to form a numerator involving a product of two terms involving y and a denominator involving a product of two terms involving $r+y$ ***(M1)***

$$\frac{y(y-1)}{(r+y)(r+y-1)}=\frac{1}{3}$$ ***A1***

attempts to remove the fractions and expand the brackets ***(M1)***

$3y^2-3y=y^2+2ry-y+r^2-r$ ***A1***

$2y^2-2ry-2y+r-r^2=0$

$2y^2-2(r+1)y+r-r^2=0$ ***AG***

[4 marks]

(b) attempts to solve for y ***(M1)***

$$y=\frac{2(r+1)\pm\sqrt{4(r+1)^2-8(r-r^2)}}{4}$$ ***A1***

$$y=\frac{2(r+1)\pm\sqrt{12r^2+4}}{4}$$ ***A1***

$$y=\frac{(r+1)\pm\sqrt{3r^2+1}}{2}$$

(since $r, y\in\mathbb{Z}^+$) and $\frac{(r+1)-\sqrt{3r^2+1}}{2}<0$ for $r>1$ ***R1***

Note: Award the ***R1*** for stating that number of balls cannot be negative, or similar.

Note: Accept $y>0$

so $y=\frac{(r+1)+\sqrt{3r^2+1}}{2}$ ***AG***

[4 marks]

continued…

Question 11 continued

(c) attempts to find a pair of positive integer values *eg* by using a table ***(M1)***

Note: Award ***M0*** if numbers are not positive integers.

1 red ball and 2 yellow balls ($r = 1$ and $y = 2$) ***A1***

4 red balls and 6 yellow balls ($r = 4$ and $y = 6$) ***A2***

Note: Award ***A1*** for one solution and ***A2*** for another.
15 red balls and 21 yellow balls ($r = 15$ and $y = 21$) is the next solution.

[4 marks]

(d) attempts to form a numerator involving a product of three terms involving y and a denominator involving a product of three terms that includes a $(y+10)$ term ***(M1)***

$$\mathrm{P}(YYY) = \frac{y(y-1)(y-2)}{(y+10)(y-1+10)(y-2+10)}\left(= \frac{y(y-1)(y-2)}{(y+10)(y+9)(y+8)}\right)$$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator.

[3 marks]

(e) $$\mathrm{P}(\text{new } YYY) = \frac{(y+1)(y)(y-1)}{(y+1+10)(y+10)(y-1+10)}\left(= \frac{(y+1)(y)(y-1)}{(y+11)(y+10)(y+9)}\right)$$ ***(A1)***

equates their answer for $\mathrm{P}(\text{new } YYY)$ to $2\times$ their answer for part (d) ***M1***

$$\frac{2y(y-1)(y-2)}{(y+10)(y+9)(y+8)} = \frac{(y+1)(y)(y-1)}{(y+11)(y+10)(y+9)}$$

attempts to solve for y ***(M1)***

Note: Award ***(M1)*** for attempting to write the above expression as
$$\frac{2(y-2)}{y+8} = \frac{y+1}{y+11}.$$

$y = 4$ ***A2***

[5 marks]

Total [20 marks]

12. (a) attempts to use $y_1 = y_0 + h \times f\left(x_0, y_0\right)$ ***(M1)***

$y_1 = 2 + 0.1 \times \dfrac{1^2 + 3\left(2\right)^2}{2}$

$= 2.65$ ***A1***

Note: Award ***(M1)A0*** for 2.35.

[2 marks]

(b) let $y = vx$ ***M1***

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = v + x\dfrac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

$v + x\dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{x^2 + 3v^2x^2}{vx^2}$ ***(M1)***

$x\dfrac{\mathrm{d}v}{\mathrm{d}x} = \dfrac{1 + 2v^2}{v}$ ***A1***

attempt to separate variables x and v ***M1***

$\int \dfrac{v}{2v^2 + 1}\mathrm{d}v = \int \dfrac{1}{x}\mathrm{d}x$

$\dfrac{1}{4}\ln\left(2v^2 + 1\right) = \ln x + C$ ***A1***

Note: Condone the absence of C to this stage.

continued…

Question 12 continued

EITHER

$$\frac{1}{4}\ln\left(\frac{2y^2}{x^2}+1\right)=\ln x+C$$

when $x=1,\ y=2\Rightarrow C=\frac{1}{4}\ln 9$ ***M1***

Note: Award ***M1*** for attempting to find their value of C.

$$\frac{1}{4}\ln\left(\frac{2y^2}{x^2}+1\right)=\ln x+\frac{1}{4}\ln 9$$

$$\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}}=\sqrt{3}x$$

OR

$$\ln\left(\frac{2y^2}{x^2+1}\right)=\ln\left(x^4\right)+\ln C$$

$$\frac{2y^2}{x^2}+1=Cx^4$$

when $x=1,\ y=2\Rightarrow C=9$ ***M1***

THEN

$$\frac{2y^2}{x^2}+1=9x^4$$ ***A1***

$$y=x\sqrt{\frac{9x^4-1}{2}}$$ ***AG***

[8 marks]

continued…

Question 12 continued

(c) $y = 2.71422...$

$y = 2.71$ ***A1***

[1 mark]

(d) **EITHER**

the graph of $y = x\sqrt{\frac{9x^4 - 1}{2}}$ is concave up (for $1 \le x \le 1.1$) ***A1***

OR

$\frac{d^2y}{dx^2} > 0$ (for $1 \le x \le 1.1$) ***A1***

Note: Allow positive curvature, opening upwards, increasing first derivative.

THEN

hence the tangent drawn using Euler's method gives an underestimate of the true value, so the value of y when $x = 1.1$ is greater than the approximate value found in part (a) ***R1***

Note: Only award ***R1*** if there is reference to tangent (in words or in a diagram)***.***

[2 marks]

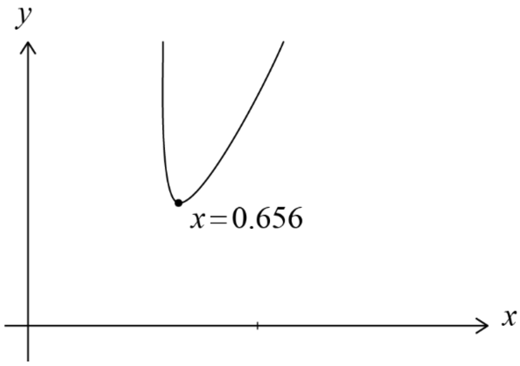
continued…

Question 12 continued

(e) **EITHER**

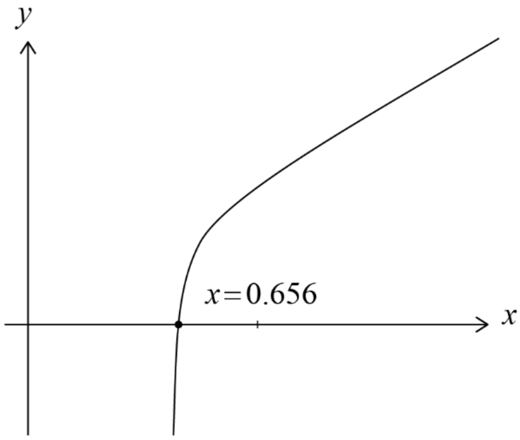
Note: Award the first ***A*** mark for a correct graph seen in part (d).

correct graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ ***A1***



OR

correct graph of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***A1***



THEN

$x = 0.655996...$

$x = 0.656$ ***A1***

[2 marks]

continued…

Question 12 continued

(f) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ (seen anywhere) ***(A1)***

Note: Award ***(A1)*** for equivalent answers (seen anywhere) such as $\frac{-x^4+x^2y^2+6y^4}{x^2y^3}=0$ or $-x^4+x^2y^2+6y^4=0$.

EITHER

divides $-x^4+x^2y^2+6y^4(=0)$ through by y^4 ***(M1)***

$-\frac{x^4}{y^4}+\frac{x^2}{y^2}+6(=0)$

$m=\frac{y}{x}\Rightarrow-\frac{1}{m^4}+\frac{1}{m^2}+6(=0)$ ***(M1)***

$6m^4+m^2-1(=0)$

$(3m^2-1)(2m^2+1)=0$ ***A1***

$m=\pm\frac{1}{\sqrt{3}}\ \left(m^2=-\frac{1}{2}\right)$

OR

divides $-x^4+x^2y^2+6y^4(=0)$ through by x^2y^2 ***(M1)***

$-\frac{x^2}{y^2}+1+6\frac{y^2}{x^2}(=0)$

$m=\frac{y}{x}\Rightarrow-\frac{1}{m^2}+1+6m^2(=0)$ ***(M1)***

$6m^4+m^2-1(=0)$

$(3m^2-1)(2m^2+1)=0$ ***A1***

$m=\pm\frac{1}{\sqrt{3}}\ \left(m^2=-\frac{1}{2}\right)$

continued…

Question 12 continued

OR

attempts to factorize $-x^4 + x^2y^2 + 6y^4 (= 0)$ ***(M1)***

$-(x^2 - 3y^2)(x^2 + 2y^2)(= 0)$ ***A1***

attempts to solve their factorized equation ***(M1)***

$\Rightarrow y = \pm\frac{1}{\sqrt{3}}x \left(y^2 = -\frac{1}{2}x^2\right)$

THEN

$y = \frac{1}{\sqrt{3}}x$ (and so $m = \frac{1}{\sqrt{3}}$) ***A1***

as $x > 0,\ y > 0$ (or equivalent reasoning/justification) ***R1***

Note: Award ***R1*** for $y = \frac{1}{\sqrt{3}}x$ (and so $m = \frac{1}{\sqrt{3}}$) as $y \neq -\frac{1}{\sqrt{3}}x$ and $x^2 + 2y^2 = 0$ for $x = 0$ and $y = 0$ only.

[6 marks]

Total [21 marks]